Arika Khor
Math 2934

In Class activity

This Exam will be on March 13th at the beginning of class time, 10:30 am until 11:45 am. You are allowed a non-graphing calculator without CAS (computer algebra system). You are also allowed both sides of a standard sized printer paper (like all the worksheets) as an equation sheet. This sheet must be handwritten (with pen or pencil). It cannot be typeset or have printed equations. Your equation sheet will be turned in with the exam, so it must have your name in the top left corner.

1) Plot the following regions using gradients (you are expected to be able to plot all these (minus perhaps (b)) by hand); What is the boundary ∂R? Are they open? Are they closed? Are they bounded?

(a) $x^2 + y^2 < 2$ (Disk)

(b) $x^2 - y^2 + 1 < 2$

(c) $2x + y \leq 2$ (Half plane)

(d) $x - 2y \leq 2$ (Another half plane

- Bounded? If $\leq$ or $\geq$ bounded, else if $<$ or $>$ then bounded.
- Open?
- Closed? Boundary included?
- Neither? Some included and some are missing frm the func

2) Write the following regions as inequalities or intersection of inequalities:

(a) The disk of radius 3 centered on $(1,1)$ $\boxed{(x-1)^2 + (y-1)^2 \leq 9}$

(b) the region $[0,1] \times [1,3]$ Simply integral bounds: $\boxed{0 \leq x \leq 1 \text{ and } 1 \leq y \leq 3}$

(c) The region below (in 2D, down is the negative y direction) the line $x - 2y = 1$

(d) The triangle with vertices $(-1,-1), (1,0)$, and $(0,1)$ Rewrite to obtain 3 lines for each pt:

$x \leq 1$

$y \geq 0$

$y \leq -x + 1$

Then take

3) For each of the following regions: Plot R, find points of intersection, x-min, y-min, x-max, y-max, a generic fixed x slice, and a generic fixed y slice:

(a) R_1 is the region between $x + 2y = 1$ and $y^2 + 2y = x$ Find y-int

$$y^2 + 2y = 1 - 2y \qquad y = -2 \pm \sqrt{5}$$

Can then find x-int by

$$Andfortheothercase($$

-$\frac{1}{\sqrt{2}}$) is $\frac{1}{2} - \frac{1}{\sqrt{2}}$

Hence results for POI are

$$(\frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \text{ and } (\frac{1}{2} - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Extrema, look at pts + take derative of either EQ

$$x(y) = y^2 + 2y \qquad x'(y) = 2y + 2 \qquad x'(y) = 0, y = -1$$

Thus xMin at -1, and the rest correspond 2 graph

To find x-slice, consider when $x \leq$ x-int cords (in this case there are 2 times this happens)

- Case 1: $x \leq 5 - 2\sqrt{5}$

  Then take $y^2 + 2y = x$
- Case 2: $x \geq 5 - 2\sqrt{5}$

  $y^2 + 2y = x$

(b) R_4 is the triangle with vertices $(-2,1)$, $(1,0)$, and $(1,3)$

Find bounds for generalized double $\int$'s. Steps for general regions:

- POI: Set sys of EQ: set both EQ = to e/o (aft putting in terms of 1 var)
- Take solution frm sys of EQ, plug into both EQ $\rightarrow$ solution points.
- Look at given points and decide what is min/max for each val

Steps for triangular regions:

- min/max use pts given
- make 3 lines
- y/x slice use points given the 3 lines:
-

4) Let $f(x,y) = 2x + 3xy^2$ approximate the change in f when $f = 10$, $y = 2$, $\Delta x = .5$, and $\Delta y = .2$.

Asking us to find total differential:

$$df = f_x \Delta x + f_y \Delta y$$

Need to find x, $10 = 2x + 3xy^2$ then $x = \frac{5}{7}$

Then finding partials: $f_x = 2 + 3y^2 \qquad f_y = 6xy$

Eval partials at $(\frac{5}{7}, 2)$, for x: 14, y: 12 (partials)

Plugging into formula we then get:

$$df = (14)(.5) + (12)(.2) = \boxed{10.36}$$

5) Let $f(x,y) = 2x^2 + y^2$ and let $g(x,y) = \dfrac{1}{x^2+y^2+1}$. Let $P_1 = (1,2)$ and $P_2 = (0,0)$. We use direction of tangent plane formula:

$$z = f(x_0,y_0) + f_x(x_0,y_0) + ... \qquad z = g(1,2) + g_x(1,2) + g_y(1,2)$$

(per x,y,z) where f_x= partial of f at those pts

- Find partials
- Find g(1,2)
- Plug into EQ: $\boxed{x + 2y + 18z = 8}$

(a) What is the equation of the tangent plane of g at P_1

6) What is the directional derivative of $f(x,y) = 3x^2 + 2xy^2$ at $P = (1,1)$ in the direction of $\hat{v} =< 1, -1 >$

directional derivative formula:

$$\boxed{\nabla f(1,1) \cdot \frac{\vec{v}}{||\vec{v}||}}$$

Mag of $\vec{v} = \sqrt{2}$ then js put that on bottom to get: $\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

7) Find all the local maxima/minima/saddlepoints of the functions:

Conditions of higher dimension relative extrema:

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b]^2$$

Where a,b are $\nabla f =< 0,0 >$, where a = solution to $\hat{i}$ and b is to (as xy pairs)

- $D > 0$?
  - $f_{xx}(a,b) > 0$? rel min at (a,b)

- $f_{xx}(a,b) < 0$? rel max at (a,b)
- $D < 0$ Saddle Point
- $D = 0$ Cry :(

So for each problem we simply find each of the required inputs for the formula.

(a) $f_1(x,y) = x^2 + y - xy$

- $f_{xx} = 2 \quad f_{xy} = -1 \quad f_{yy} = 0 \quad f_{xy} = -1$
- Then our a,b is $< 2-y, 1-x > = < 0, 0 >$ or $(1,2)$
- Then simply plug into EQ

8) Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2+1}}$; optimize f on $2x^2 + y^2 = 1$.

We find the gradient of both equations, solve for the systems of equations with λ, then plug into the constraints to obtain the min and max.

- $\nabla f(P) = < -\frac{x}{(x^2+y^2+1)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+1)^{\frac{3}{2}}} >$
- $\nabla g(P) = < 4x, 2y >$
- Setting up SoE:

$$-\frac{x}{(x^2+y^2+1)^{\frac{3}{2}}} = \lambda 4x \qquad -\frac{y}{(x^2+y^2+1)^{\frac{3}{2}}} = \lambda 2y$$

Solving for one case we have:

$$\frac{x}{4x} = \frac{y}{2y}, \qquad x = 0, y = 0$$

For the first sub-case we have:

$$y^2 = 1, y = \pm 1$$

Hence we have 2 solutions so far:

$$(0,1), (0,-1)$$

Moving onto the next set $2x^2 = 1, x = \pm\frac{1}{2}$ Then we have the following solutions:

$$(\frac{1}{\sqrt{2}}, 0), (-\frac{1}{2}, 0)$$

We then compute each point $(0,1), (0,-1), (\frac{1}{\sqrt{2}}, 0), (-\frac{1}{2}, 0)$ at ∇f:

$$-(0,1), (0,-1) \rightarrow (0, \frac{1}{2\sqrt{2}}), \lambda = \frac{1}{4\sqrt{2}}$$

9) Evaluate the following integrals:

(a) $\iint_R x^2y + xy + 1 dA$; $R = [0,1] \times [0,1]$ (the answer is $\frac{17}{12}$) Same deal:

$$\int_0^1 \int_0^1 x^2y + xy + 1 dx dy$$

Finding inner integral by splitting then integrating each term:

$$\int_0^1 x^2y + xy + 1 dx = x + \frac{yx^2}{2} + \frac{yx^3}{3}]_0^1 = 1 + \frac{y}{2} + \frac{y}{3}$$

Solve for outside using same method as first time:

$$\int_0^1 1+\frac{y}{2}+\frac{y}{3}dy = y+\frac{y^2}{4}+\frac{y^2}{6}]_0^1 = 1+1/4+1/6 = \boxed{\frac{17}{12}}$$

(b) $\iint_R x^2+y^2+1dA$; R is the region between $y=1-x^2$ and $y=x^2-1$. (the answer is $\frac{80}{21}$) We are tasked with finding a double integral bounded by two general regions. We first need to find out bounds for the integral by setting the regions equal to each other:

$$1-x^2 = x^2-1 \qquad x=\pm 1$$

Notice how we have now have 2 possible answers, however this doesn't matter given both input regions are to the power of some even term (actually what if it did then what). We can take a vertical slice to obtain (-1,1) for our bounds and solve like normal now:

$$\int_{-1}^1 \int_{x^2-1}^{1-x^2} x^2+y^2+1dydx$$

We solve the inside by splitting each term and cancelling the $2x^2$:

$$\int_{x^2-1}^{1-x^2} x^2+y^2+1dy = \int_{x^2-1}^{1-x^2} y+x^2y+\frac{y^3}{3} = -2x^4+\frac{-2x^6+6x^4-6x^2+2}{3}+2$$

Then taking the outside integral term by term:

$$\int_{-1}^1 -2x^4+\frac{-2x^6+6x^4-6x^2+2}{3}+2 = -\frac{2x^7}{21}-\frac{2}{3}x^3+\frac{8}{3}x]_{-1}^1 = -\frac{4}{5}+\frac{64}{105}+4 = \boxed{\frac{80}{21}}$$

10) Let $f(x,y)=2x^2+xy$. Integrate f over some different region:

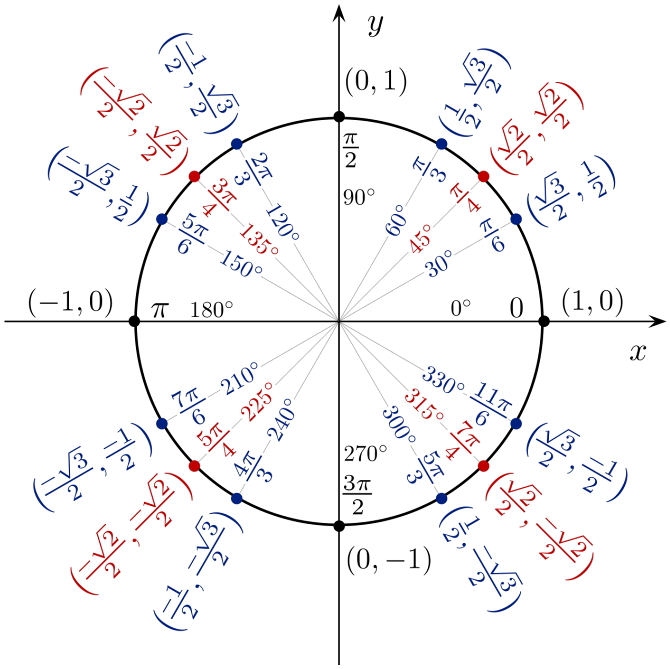
Use $f(x,y)$ for inside. Outside: Use diff regions outside for each

$$\int_{-1}^1 \int_{-x}^{\frac{5+2x}{3}}$$

For some $R=[a,b]\times[c,d]$

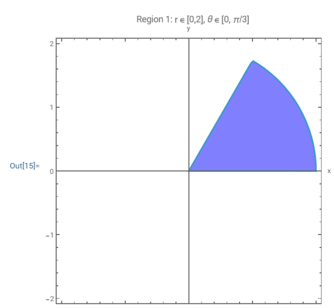
11) Plot the following regions: First interval: Radius (dst frm origion, start at -1 -> go to 1) Second interval: Direction frm origion (How much of the circle spans)

Tldr use unit circle, first pt = x-axis, second pt= whatever π in unit circle.



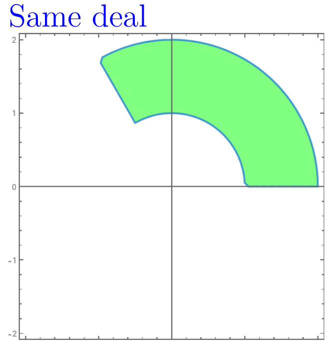
(a) $(r, \theta) \in [0, 2] \times [0, \pi/3]$

Origin pt @ 0 -> x=2 for second part $[0, \pi/3]$, 0 starts at y=0, extends up to $\pi/3$



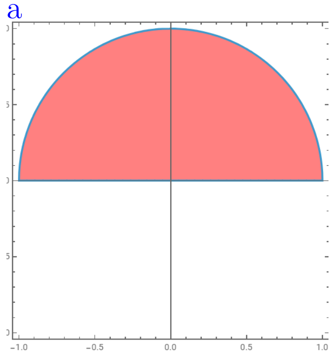
(b) $(r, \theta) \in [1, 2] \times [0, 2\pi/3]$

Same deal



(c) $(r, \theta) \in [-1, 1] \times [0, \pi]$

a



12) Evaluate the following integrals:

(a) $\iint_R x^2 + y^2 + x \; dA$, where R is the unit circle centered on the origin

- Convert to polar:

$$x = rcos\theta \quad y = rsin\theta \quad dA = rdrd\theta$$

Knowing that $x^2 + y^2 = r^2 \quad x = rcos\theta$, so we have:

$$\int_0^{2\pi} \int_0^1 (r^2 + rcos\theta) rdrd\theta$$

First $\int$ solution: $\frac{1}{4} + \frac{1}{3}cos\theta$ then solve for second $\int$

(b) $\iint_R x^2 - y^2 \; dA$, where R is the annulus (this is a good word to know) with inner radius 1 and outer radius 2

- In this case simply convert for inside to polar:

$$x^2 - y^2 = r^2(cos^2\theta - sin^2\theta)$$

13) Show that the function $f(x, y) = |xy|$ is not differentiable.

This is the Euclidean norm. To prove, check the partials of both x,y and eval both partials at $(0, 0)$. Take the limit of each partial, where $\lim_{(x,0)\to(0,0)} \partial x$. In this case, taking lim results in

$$lim_{x\to 0} \frac{x}{|x|}$$

So its not differentiable.

14) Miscellaneous Knowledge:

(a) I recommend reviewing your trig integrals as they often show up in polar integration

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

(b) I recommend working out the basic properties (partial derivatives, maxes/mins,...) of $f(x, y) = e^{-x^2-y^2}$ (the 2D Gaussian distribution without its normalizing factor)

Find both partials, set both equal to 0. To find critical pts, find second-order partials (see q7). In this case global and rel are the same

(c) max: $(0, 0)$

(d) No min, but $\lim_{x,y\to\infty} f(x, y) = 0$

(e) I recommend working out the basic properties of $f(x, y) = -\frac{1}{\sqrt{x^2 + y^2}}$ (the potential function of 2/4 of the fundamental forces)

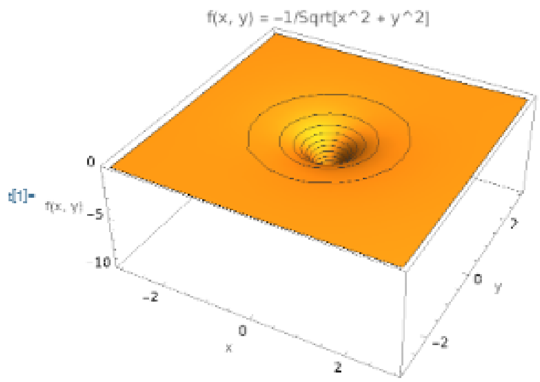
No rel extrema

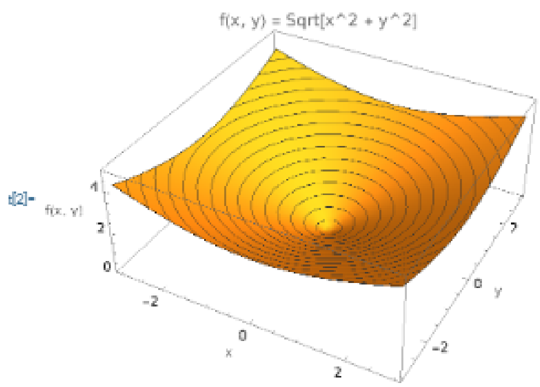
Tends towards $-\infty$ near origin, approaches 0 as (x,y) $\rightarrow \infty$

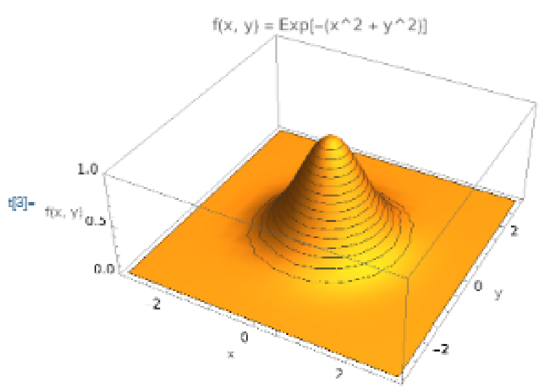
(f) I recommend working out the basic properties of $f(x, y) = \sqrt{x^2 + y^2}$ (the distance function)

- Rel min at $(0, 0)$ with $f(0, 0) = 0$
- Increase as $(x, y) \rightarrow \infty$
- Global max = $(0, 0)$ no rel max, no upper bound

Rel min at $(0, 0)$ with $f(0, 0) = 0$ Increase as $(x, y) \rightarrow \infty$, no rel max







(g) Polar Cords rep

$$x = rcos\theta \qquad y = rsin\theta$$

Where

$$r = \sqrt{x^2 + y^2} \qquad \theta = tan^{-1}\frac{y}{x}$$

(h) Steps for polar integration: